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Connected Hyperplanes in Binary Matroids

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In this paper, we prove that any simple and cosimple connected binary matroid has at least four connected hyperplanes. We further prove that each element in such a matroid is contained in at least two connected hyperplanes. Our main result generalizes a matroid result of Kelmans, and independently, of Seymour. The following consequence of the main result generalizes a graph result of Thomassen and Toft on induced non-separating cycles and another graph result of Kaugars on deletable vertices. If G is a simple 2-connected graph with minimum degree at least 3, then, for every edge e, there are at least two induced non-separating cycles avoiding e and two deletable vertices non-incident to e. Moreover, G has at least four induced non-separating cycles.

1. INTRODUCTION

Some connected matroids have no connected hyperplanes. This is true, for example, for connected uniform matroids $U_{r,n}$ where $n > r > 2$. It is somewhat striking that every 2-connected binary matroid which is simple and cosimple has at least one connected hyperplane (Kelmans and independently, Seymour—see Theorem 1.2 below). We strengthen this result to show every 2-connected binary simple cosimple matroid has at least four connected hyperplanes. We further prove that each element in such a matroid is contained in at least two connected hyperplanes. In Section 3, we will show that these results are best possible by giving an infinite family of graphs which attain these bounds.

For matroid notation, we follow Oxley [13]. We let $M(G)$ represent the cycle matroid corresponding to the graph G. A matroid M is *simple* if M

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has no circuit of size less than 3; M is *cosimple* if the dual matroid, M^*, is simple. Thus, a matroid is cosimple if M has no cocircuit of size less than 3. A matroid M is *connected* if every two distinct elements of M are contained in some common circuit. A *loop* in a matroid is a single-element circuit; a *coloop* is a single-element cocircuit.

Let M be a matroid and k be a positive integer, a k-separation of M is a partition $\{X, Y\}$ of $E(M)$ such that

$$\min\{|X|, |Y|\} \geqslant k \tag{1}$$

and

$$r(X) + r(Y) - r(M) \leqslant k - 1. \tag{2}$$

For all $n \geqslant 2$, M is *n-connected* if, for all k in $\{1, 2, ..., n-1\}$, M has no k-separation. A matroid M is 2-connected if and only if M is connected. For the rest of the paper, we will say a matroid is connected if it is 2-connected.

Let M be a connected matroid. A circuit C of M is *non-separating* if the contraction M/C is connected. A cocircuit C^* of M is *non-separating* if the contraction M^*/C^* and hence the deletion $M \backslash C^*$ is connected. A cycle of a connected graph G is *non-separating* if $G - V(C)$ is connected. Let G be a simple 2-connected graph with minimum degree at least 3. Then it is straightforward to verify that C is a non-separating circuit of $M(G)$ if and only if C is an induced non-separating cycle of G. We will use this fact frequently in the paper.

Let M be a connected matroid. A hyperplane H of M is *connected* if the restriction $M \mid H$ is a connected matroid. Therefore, a cocircuit C^* of M is non-separating if and only if $E(M) - C^*$, the corresponding hyperplane, is connected. Let G be a simple 2-connected graph. A cocircuit of $M(G)$ corresponds to a minimal edge cut of G. Thus, a non-separating cocircuit of $M(G)$ corresponds to the set of edges incident to some vertex v such that $G \backslash v$ is 2-connected. Such a vertex is called a *deletable* vertex. Let $M = M^*(G)$, where G is simple 2-connected with minimum degree at least 3. Then a non-separating cocircuit of M is exactly an induced non-separating cycle in the graph G. Non-separating circuits and cocircuits are very useful in studying the structure of graphic matroids (see, for example, Kelmans [9, 10], Thomassen and Toft [14], and Tutte [15]).

There has been much interest in the study of deletable cycles [4, 7], induced non-separating cycles [14, 15] for graphs, non-separating cocircuits for graphs and matroids [1, 8, 10], and, recently, deletable circuits in connected matroids [5, 11]. The concept of non-separating cocircuits in matroids is closely related to induced non-separating cycles and deletable cycles in graphs and deletable circuits in matroids. In this paper we study connected hyperplanes and non-separating cocircuits in binary matroids. We give best-possible lower bounds on the number of connected hyperplanes and non-separating cocircuits for simple, cosimple, connected binary matroids. We first state several graph and matroid results which are related to our main theorem.

Kaugars (in [6]) proved that certain graphs have at least one deletable vertex and hence at least one non-separating cocircuit.

THEOREM 1.1 (Kaugars). *Let G be a simple 2-connected graph. If each vertex has degree at least three, then G has at least one deletable vertex.*

Lozovanu and Syrbu (in [16, p. 102]) improved Kaugars' bound by showing that such a graph has at least four deletable vertices. Kelmans [8] and, independently, Seymour (in [12]) proved the following binary matroid version of Kaugars' result.

THEOREM 1.2 (Kelmans; Seymour). *Let M be a simple and cosimple connected binary matroid. Then M has at least one non-separating cocircuit.*

Theorem 1.2 was conjectured by Thomassen and Toft [14] motivated by the following result:

THEOREM 1.3 (Thomassen and Toft). *Let G be a simple 2-connected graph with minimum degree at least three. Then G contains an induced non-separating cycle.*

Let G be a simple 2-connected graph with minimum degree at least three. Then a cycle C is induced and non-separating in G if and only if C is a non-separating cocircuit in the matroid $M^*(G)$. Thus we can view Theorem 1.2 as a simultaneous generalization of Theorems 1.1 and 1.3. Note that, Theorem 1.2 can be restated as *a simple and cosimple connected binary matroid has a connected hyperplane*. Our main theorem, the next result, and its corollary give generalizations of Theorem 1.2 and the graph result of Lozovanu and Syrbu. Proofs of these results will be delayed until Section 2. In Section 3, we will give some consequences which generalize Theorems 1.1 and 1.3.

THEOREM 1.4. *Let M be a connected simple and cosimple binary matroid. Then every element is in at least two connected hyperplanes. Furthermore, M has at least four connected hyperplanes.*

Theorem 1.4 can be restated in terms of cocircuits, since an element e is in a connected hyperplane of a matroid if and only if e avoids a non-separating cocircuit. In addition, if M is a simple, cosimple, connected binary matroid, then M^* is also a simple, cosimple, connected binary matroid. Thus, we also get a circuit version of Theorem 1.4.

COROLLARY 1.5. *Let M be a connected simple and cosimple binary matroid. Then, for every element e,*

(i) *there are at least two non-separating cocircuits avoiding e, and*

(ii) *there are at least two non-separating circuits avoiding e.*

Moreover, M has at least four non-separating cocircuits and at least four non-separating circuits.

2. A PROOF OF THE MAIN RESULT

In this section, we give a proof of our main theorem. We begin with several preliminary results needed in the proof. The first result (found, for example, in [13]) gives information about circuit sizes of an n-connected matroid.

LEMMA 2.1. *If M is an n-connected matroid and $|E(M)| \geqslant 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

We use $\mathscr{C}(M)$ to denote the circuit set of a matroid M. Let $M_1 = (E_1, \mathscr{C}(M_1))$, and $M_2 = (E_1, \mathscr{C}(M_2))$ be two matroids with $E_1 \cap E_2 = \{p\}$. Suppose p is neither a loop or nor a coloop of M_1 or M_2. The *parallel connection $P(M_1, M_2)$* (Brylawski [2]) of M_1 and M_2 with respect to the basepoint p is defined as the matroid whose circuit set is

$$\mathscr{C}_P = \mathscr{C}(M_1) \cup \mathscr{C}(M_2) \cup \{(C_1 - p) \cup (C_2 - p) : p \in C_i \in \mathscr{C}(M_i) \text{ for } i = 1, 2\}.$$

If p is not a separator of M_1 or M_2, then the 2-*sum* of M_1 and M_2, denoted by $M_1 \oplus_2 M_2$, is the matroid $P(M_1, M_2) \backslash p$. The next three lemmas can be found in [13].

LEMMA 2.2. *If $e \in E(M_1) - p$, then $P(M_1, M_2) \backslash e = P(M_1 \backslash e, M_2)$.*

LEMMA 2.3. *Given the matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$. Then*

(i) $(M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^*$.

(ii) *if $|E(M_i)| \geqslant 2$ for $i = 1, 2$, then $P(M_1, M_2) \backslash p$ is connected if and only if both M_1 and M_2 are connected. In particular, $M_1 \oplus_2 M_2$ is connected if and only if both M_1 and M_2 are connected.*

Let M be a 2-connected but not 3-connected matroid and suppose that (X_1, X_2) is a 2-separation of M. Let p be an element that is not in $E(M)$. For $i=1, 2$, define the matroids M_i on the ground set $X_i \cup p$ with circuit-set $\mathscr{C}_i = \mathscr{C}(M \mid X_i) \cup \{(C \cap X_i) \cup p : C$ is a circuit of M that meets both X_1 and $X_2\}$. The matroids M_1 and M_2 will be called the *matroids associated with the* 2*-separation* (X_1, X_2).

Lemma 2.4. *Let M be a 2-connected but not 3-connected matroid such that M_1 and M_2 are the matroids associated with the 2-separation (X_1, X_2). Then $M = M_1 \oplus_2 M_2$.*

Lemma 2.5. *Let $M = M_1 \oplus_2 M_2$ be a simple matroid. Suppose that C^* is a cocircuit of M_1 avoiding the basepoint p. Then*

(i) *C^* is also a cocircuit of M;*

(ii) *Suppose that M is connected and binary. If C^* is a non-separating cocircuit of M_1 where $|E(M_1)| \geqslant 4$, then C^* is also a non-separating cocircuit of M.*

Proof. (i) As C^* is a cocircuit of M_1, it is circuit of M_1^*. Since $p \notin C^*$, we deduce that C^* is a circuit of M^* and thus a cocircuit of M.

(ii) By [13, Proposition 7.1.19], M_1 is a binary matroid. Now, $M \backslash C^* = (M_1 \oplus_2 M_2) \backslash C^* = (P(M_1, M_2) \backslash p) \backslash C^* = (P(M_1, M_2) \backslash C^*) \backslash p$. By Lemma 2.2, $(P(M_1, M_2) \backslash C^*) \backslash p = P(M_1 \backslash C^*, M_2) \backslash p = (M_1 \backslash C^*) \oplus_2 M_2$. By Lemma 2.3, both M_1 and M_2 are connected, since M is connected. Now, as C^* is a non-separating cocircuit in M_1, the matroid $M_1 \backslash C^*$ is connected. If the hyperplane $E(M_1) \backslash C^*$ of M_1 has at least two elements, then, by Lemma 2.3 again, $M \backslash C^*$ is connected and therefore C^* is a non-separating cocircuit of M. Suppose $E(M_1) \backslash C^*$ has exactly one element (which must be p). Then M_1 is a rank-2 binary matroid. As $\{p\}$ is a hyperplane of M_1, we deduce that p is not in any 2-element circuit of M_1. As M is simple, we conclude that M_1 is also simple. Therefore M_1 has at most three elements; a contradiction to our assumption. We conclude that $E(M_1) \backslash C^*$ has at least two elements and thus C^* is a non-separating cocircuit of M. ▌

Let (X_1, X_2) be a 2-separation of a simple matroid M. We say that X_1 is *minimal* if M has no 2-separation (Y_1, Y_2) such that Y_1 is a proper subset of X_1. The proof of the following result is straightforward and will be omitted.

Lemma 2.6. *Let (X_1, X_2) be a 2-separation of a connected matroid M and let M_1 and M_2 be the matroids associated with the 2-separation. If X_1 is minimal, then M_1 is 3-connected.*

COROLLARY 2.7. *Let M be a connected simple and cosimple binary matroid and let (X_1, X_2) be a 2-separation of M. Then $|X_1|, |X_2| \geqslant 5$. Moreover, $|X_1| = 5$ if and only if $M | X_1 \cong M(K_4) \backslash p$.*

Proof. We may assume that $|X_1|$ is minimal. As M is both simple and cosimple, $|X_1| \geqslant 3$. By Lemma 2.6, M_1 is 3-connected. If $|X_1| = 3$ or 4, then M_1 has four or five elements and thus is isomorphic to $U_{2,4}$, $U_{2,5}$, or $U_{3,5}$. This is a contradiction as M, and hence M_1, is binary. We conclude that $|X_1| \geqslant 5$. Similarly, we can show that $|X_2| \geqslant 5$. If $|X_1| = 5$, then, by Lemma 2.6 again, M_1 is a 3-connected matroid with six elements. As M is binary, by [13, p. 294], $M_1 \cong M(K_4)$. ∎

The next result is due to Bixby and Cunningham [1], and independently, Kelmans (see [10]).

LEMMA 2.8. *Let M be a 3-connected binary matroid on E, where $|E| \geqslant 4$, and let $e \in E$. Then e is an element of two distinct non-separating cocircuits of M. Moreover, M is graphic if and only if each element e is in at most two non-separating cocircuits.*

Let C^* be a cocircuit of a connected matroid M. A *bridge* of C^* in M is a component of the matroid $M \backslash C^*$ [1].

LEMMA 2.9 (Bixby and Cunningham). *Let M be a 3-connected binary matroid with at least four elements. Let C^* be a separating cocircuit containing x, and B be a bridge of C^*. Then there exists a cocircuit C_1^* such that $x \in C_1^*$ and $B \subset B_1$ for some bridge B_1 of C_1^*.*

We break the proof of Theorem 1.4 into several lemmas. We first consider the case where M is 3-connected.

LEMMA 2.10. *Let M be a 3-connected binary matroid with at least four elements. Then*

(i) *for any distinct elements e and f of M, there is a connected hyperplane containing e and avoiding f,*

(ii) *for any element e, there are connected hyperplanes $H_1, H_2, ..., H_k$ for some $k \geqslant 2$ such that $H_1 \cap H_2 \cap \cdots \cap H_k = \{e\}$.*

Proof. (i) Let e and f be two distinct elements of M. As M is 3-connected with at least four elements, the matroid M/e is connected having at least three elements. Thus M/e has a cocircuit C_1^* which contains f (and avoids e). Clearly, C_1^* is also a cocircuit of M which contains f and avoids e. As M is 3-connected and $|E(M)| \geqslant 4$, it follows from Lemma 2.1 that $|C_1^*| \geqslant 3$. Let B_1 be the bridge of C_1^* containing e. By Lemma 2.9, there exists a cocircuit C_2^* such that $f \in C_2^*$ and $B_1 \subset B_2$ for some bridge B_2 of C_2^*. Continuing the process if necessary, we obtain a non-separating cocircuit J and bridge $E-J$ such that $f \in J$ and $B_1 \subseteq E-J$. As $e \in B_1 \subseteq E-J$, the connected hyperplane $E-J$ contains e and avoids f, as required.

(ii) Let H_1 be a connected hyperplane containing e. As $|E(M)| \geqslant 4$ and M is binary, $r(M) \geqslant 3$. Thus H_1 has at least two elements. For each element f in $H_1 \backslash e$, by (i), there is a connected hyperplane containing e and avoiding f. The intersection of all these hyperplanes is equal to $\{e\}$, as required. ∎

Corollary 2.11. *Let M be a 3-connected binary matroid having at least four elements. Then M has at least four non-separating cocircuits. Moreover, if M has exactly four non-separating cocircuits, then $M \cong M(K_4)$.*

Proof. First we assume that M is not graphic. By Lemma 2.8, there is an element e which is contained in at least three non-separating cocircuits. By Lemma 2.10, e avoids at least two non-separating cocircuits. Thus, M has at least five non-separating cocircuits. If M is graphic, then $M \cong M(G)$ for some 3-connected graph G. Then, every vertex of G is deletable. Since $|V(G)| \geqslant 4$, M has at least four non-separating cocircuits. If G has exactly four non-separating cocircuits, then $|V(G)| = 4$ and $M \cong M(K_4)$. ∎

Lemma 2.12. *Let M be a connected simple and cosimple binary matroid. If (Y_1, Y_2) is a 2-separation of M, then both Y_1 and Y_2 contain at least two non-separating cocircuits.*

Proof. For the 2-separation (Y_1, Y_2), we can find a 2-separation (X_1, X_2) such that $X_1 \subseteq Y_1$ and X_1 is minimal. Next we show that X_1 contains at least two non-separating cocircuits of M. By Lemma 2.4, $M = M_1 \oplus_2 M_2$, where M_1 and M_2 are the matroids associated with X_1 and X_2. By Corollary 2.7, M_1 has at least five elements. By Lemma 2.10, there are at least two connected hyperplanes containing p in M_1. Thus, there are at least two non-separating cocircuits, denoted by C_1^* and C_2^*, of M_1 avoiding p. By Lemma 2.5, both C_1^* and C_2^* are non-separating cocircuits of M. Hence $Y_1 \supseteq X_1$ contains at least two non-separating cocircuits of M. By symmetry, Y_2 also contains at least two non-separating cocircuits. ∎

Now we are ready to begin the proof of Theorem 1.4.

Proof of Theorem 1.4. Let M be a connected simple and cosimple binary matroid. Clearly, M has at least four elements. If M is 3-connected, then by Lemma 2.10 and Corollary 2.11, the theorem holds. Thus, we may assume that M has a 2-separation (X, Y). By Lemma 2.12, X and Y each contain at least two non-separating cocircuits. Thus, M has at least four non-separating cocircuits. Let e be an element of M. We may assume that $e \in Y$. Let C_1^* and C_2^* be two non-separating cocircuits of M contained in the set X. Then $E(M)\backslash C_i^*$ $(i=1, 2)$ are two connected hyperplanes of M containing e. This completes the proof of the theorem. ∎

Corollary 1.5 is an immediate consequence of Theorem 1.4. Let G be a simple 2-connected graph with no two-element edge cut. Then $M(G)$ is simple and cosimple and thus a non-separating circuit of $M(G)$ must be an induced non-separating cycle in G. Therefore we have the following immediate consequence of Corollary 1.5.

Corollary 2.13. *Let G be a simple 2-connected graph which has no 2-element edge cut. Then, for every edge e, there are at least two induced non-separating cycles avoiding e and two deletable vertices non-incident to e. Moreover, G has at least four induced non-separating cycles and four deletable vertices.* ∎

3. SOME CONSEQUENCES

In this section, we present some consequences of the main theorem. These results generalize Theorems 1.1 of Kaugars and 1.3 of Thomassen and Toft. We also derive the graph result of Lozovanu and Syrbu from our main result.

Corollary 3.1. *Let G be a simple 2-connected graph with minimum degree at least three. Then for every edge e, there are at least two induced non-separating cycles avoiding e and two deletable vertices non-incident to e. Moreover G has at least four induced non-separating cycles and four deletable vertices.*

Proof of Corollary 3.1. Let G be a simple 2-connected graph with minimum degree at least three. Clearly, G has at least six edges. Let e be an edge of G. We use induction on the number of edges of G. If G has exactly six edges, then $G \cong K_4$ and the result holds. If G has no 2-element edge-cut, then by Corollary 2.13, the result holds. Suppose that G has a 2-element edge cut $\{s, t\}$. Without loss of generality, assume that $s \neq e$ and consider the new graph $G_1 = G/s$ (otherwise, consider $G_1 = G/t$). Assume that $t = uv$, $s = w_1 w_2$, and denote the new vertex of G_1 obtained by contracting s as w. Clearly, G_1 is simple, is 2-connected, and has minimum degree at least 3. By induction, G_1 has at least two induced non-separating cycles avoiding e and two deletable vertices non-incident to e. Moreover G_1 has at least four induced non-separating cycles and four deletable vertices. Next we show that an induced non-separating cycle C of G_1 is still an induced non-separating cycle of G. Clearly, both $\{u, w\}$ and $\{v, w\}$ are vertex-cuts of G_1. Suppose that $V(C) \cap \{u, v, w\} = \{v, w\}$. Then it is easy to see that G_1/C has a cut-vertex, a contradiction as C is non-separating in G_1. By a similar argument, we can show that $V(C) \cap \{u, v, w\}$ has at most one element. Thus C is still a cycle of G. Clearly, it is an induced circuit of G also. Therefore, G/C has no loop. Since G is 2-connected, G/C has no coloop. As $(G/C)/e = G/e/C = G_1/C$ is 2-connected, we deduce that G/C is also 2-connected. Thus C is also a non-separating cycle of G. If C avoids e in G_1, then C also avoids e in G. Hence G has at least two induced non-separating cycles avoiding e and G has at least four induced non-separating cycles. By a similar argument, we can prove that G has two deletable vertices non-incident to e and has at least four deletable vertices. ∎

The next result generalizes Theorem 1.2 for binary matroids which are connected but not 3-connected. It is a direct consequence of Lemma 2.12 and its dual and we will omit its proof.

THEOREM 3.2. *Let M be a connected simple and cosimple binary matroid. If M is not 3-connected, then M contains at least two disjoint non-separating circuits and two disjoint non-separating cocircuits.*

If we let $M = M(G)$ in Theorem 3.2, we obtain the following corollary:

COROLLARY 3.3. *Let G be a simple 2-connected graph which has no 2-element edge cut. If G is not 3-connected, then G contains at least two edge disjoint induced non-separating cycles and two non-adjacent deletable vertices.* ∎

The following corollary generalizes Theorem 1.3 for simple 2-connected graphs which are not 3-connected. It says that we can weaken the condition in the above corollary.

COROLLARY 3.4. *Let G be a simple 2-connected graph with minimum degree at least 3. If G is not 3-connected, then G contains at least two edge disjoint induced non-separating cycles and two non-adjacent deletable vertices.*

Proof of Corollary 3.4. Since G is 2-connected and not 3-connected, G has a 2-element vertex-cut. As the minimum degree of G is at least 3, it is easy to verify that G must have at least six vertices, and moreover, if G has

FIG. 1. Graphs with four induced non-separating cycles and four deletable vertices.

exactly six vertices, the corollary is true. We proceed by induction on the number of vertices of G. If G has no 2-element edge cut, then by the last corollary, the result holds. Assume that G has a 2-element edge cut $\{s, t\}$. Consider the new graph G_1 where $G_1 = G/s$. Then G_1 is simple 2-connected and has minimum degree at least three. By induction, G_1 has at least two disjoint non-separating cycles, denoted by C_1 and C_2. By the argument in the proof of Corollary 3.1, both C_1 and C_2 are induced non-separating circuits of G. Moreover, $C_1 \cap C_2 = \emptyset$. By a similar argument, we can show that G has at least two non-adjacent deletable vertices, as required. ∎

Next we give a class of simple and cosimple 2-connected graphic matroids having exactly four deletable vertices and exactly four induced non-separating cycles; see Fig. 1. This shows that the main result in Theorem 1.4 is best possible. Begin with two copies of K_4 minus an edge. Let u_0, v_0 denote the two vertices of degree 2 in the first copy, and u, v denote the two vertices of degree 2 in the second copy. Let $k \geqslant 0$. Connect u_0 and u, v_0 and v by two disjoint paths $P(u_0, u) = u_0 u_1 \cdots u_{k+1}$ where $u_{k+1} = u$, and $Q(v_0, v) = v_0 v_1 \cdots v_{k+1}$ where $v_{k+1} = v$. Add the following edges: $u_i v_i$ for all $i = 1, 2, \ldots k$ and $u_i v_{i+1}$ for all $i = 0, 1, \ldots k$. Clearly, the resulting graph G is simple, cosimple, and 2-connected. It is straightforward to show that G has exactly four deletable vertices and exactly four induced non-separating cycles.

One may ask if our main results can be generalized to other classes of matroids with higher connectivity. It appears that this is not the correct avenue to pursue, as J. Bonin (personal communication) pointed out that 3-connected ternary matroids may not have any connected hyperplanes. Take, for example, $PG(2, 3)^*$, the dual matroid of the ternary projective plane. This matroid is 3-connected, but has no connected hyperplane.

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